

On the notion of persistence of excitation for linear switched systems

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Abstract

The paper formulates the concept of persistence of excitation for discrete-time linear switched systems, and provides sufficient conditions for an input signal to be persistently exciting. Persistence of excitation is formulated as a property of the input signal, and it is not tied to any specific identification algorithm. The results of the paper rely on realization theory and on the notion of Markov-parameters for linear switched systems.

1 Introduction

The paper formulates the concept of persistence of excitation for *discrete-time linear switched systems* (abbreviated by DTLSSs). DTLSSs are one of the simplest and best studied classes of hybrid systems, [23]. A DTLSS is a discrete-time switched system, such that the continuous sub-system associated with each discrete state is linear. The switching signal is viewed as an external input, and all linear systems live on the same input-output-and state-space.

We define persistence of excitation for input signals. More precisely, we will call an input signal persistently exciting for an input-output map, if the response of the input-output map to that particular input determines the input-output map uniquely. In other words, the knowledge of the output response to a persistently exciting input should be sufficient to predict the response to **any** input.

Persistence of excitation is essential for system identification and adaptive control. Normally, in system identification the system of interest is tested only for one input sequence. One of the reason for this is that our notion of the system entails a fixed initial state. However, any experiment changes that particular initial state and it is in general not clear how to reset the system to a particular initial state. The objective is to find a

system model based on the response to the chosen input. However, the knowledge of a model of the system immediately implies that the response of the system to **any** input is known. Hence, intuitively it is clear that persistence of excitation of the input signal is a prerequisite for a successful identification of a model.

Note that persistence of excitation is a joint property of the input and of the input-output map. That is, a particular input might be persistently exciting for a particular system and might fail to be persistently exciting for another system. In fact, it is not a priori clear if any system admits a persistently exciting input. This calls for investigating classes of inputs which are persistently exciting for some broad classes of systems.

In the existing literature, persistence of excitation is often defined as a specific property of the measurements which is **sufficient** for the correctness of **some** identification algorithm. In contrast, in this paper we propose a definition of persistence of excitation which is **necessary** for the correctness of **any** identification algorithm.¹ Obviously, the two approaches are complementary. In fact, we hope that the results of this paper can serve as a starting point to derive persistence of excitation conditions for specific identification algorithms.

Contribution of the paper We define persistence of excitation for finite input sequences and persistence of excitation for infinite input sequences.

We show that for every input-output map which is realizable by a reversible DTLSS, there exists a finite input sequence which is persistently exciting for that particular input-output map. A reversible DTLSS is a DTLSS continuous dynamics of which is invertible. Such systems arise naturally by sampling continuous-time systems. In addition, we define the class of reversible input-output maps and show that there is a finite input sequence which is persistently exciting for all the input-output maps of that class. Moreover, we present a procedure for constructing such an input sequence.

We show that there exists a class of infinite input sequences which are persistently exciting for all the input-output maps which are realizable by a *stable DTLSS*. The conditions which the input sequence must satisfy is that each subsequence occurs there infinitely often (i.e. the switching signal is rich enough) and that the continuous input is a colored noise. Hence, this result is consistent with the classical result for linear systems.

It might be appealing to interpret the conditions above as ones which ensure that one stays in every discrete mode long enough and the continuous input is persistently exciting in the classical sense. One could then try to identify the linear subsystems separately and merge the results. Unfortunately, such an interpretation is in general incorrect. The reason for this is that there exists a broad class of input-output maps which can be realized by a linear switched system but not by a switched system whose linear subsystems are minimal, [20]. The above scheme obviously would not work for such systems. In fact, for such systems one has to test the system's response not only for each discrete mode, but

¹In fact, we also propose a specific algorithm for the correctness of which persistence of excitation is sufficient, but we do not claim this is true for all identification algorithms.

for each combination of discrete modes.

The main idea behind the definition of persistence of excitation and the subsequent results is as follows. From realization theory [20] we know that the knowledge of (finitely many) Markov-parameters of the input-output map is sufficient for computing a DTLSS realization of that map. Hence, if the response of the input-output map to a particular input allows us to compute the necessary Markov-parameters, then we can compute a DTLSS representation of that map. This can serve as a definition of persistence of excitation. We call a input sequence persistently exciting, if the Markov-parameters of the input-output map can be computed from the response of the map to that input. We call an infinite sequence input persistently exciting, if from a large enough finite initial part of the response one can compute an arbitrarily precise approximation of the Markov-parameters. Since the realization algorithm for DTLSS is continuous in the Markov-parameters, it means that a persistently exciting infinite input sequence allows the computation of an arbitrarily precise approximation of a DTLSS realizing the input-output map.

Motivation of the system class The class of DTLSSs is the simplest and perhaps the best studied class of hybrid systems. In addition to its practical relevance, it also serves as a convenient starting point for theoretical investigations. In particular, any piecewise-affine hybrid system can be viewed as a feedback interconnection of a DTLSS with an event generating device. Hence, identification of a piecewise-affine system is related to the problem of closed-loop identification of a DTLSS. For the latter, it is indispensable to have a good notion of persistence of excitation. For this reason, we believe that the results of the paper will be relevant not only for identification of DTLSSs, but also for identification of piecewise-affine hybrid systems with autonomous switching.

Related work Identification of hybrid systems is an active research area, with several significant contributions [26, 16, 9, 15, 17, 25, 12, 5, 13, 11, 9, 3, 2, 22, 6, 24, 18, 1]. While enormous progress was made in terms of efficient identification algorithms, the fundamental theoretical limitations and properties of these algorithms are still only partially understood. Persistence of excitation of hybrid systems were already addressed in [26, 25, 27, 24, 10]. However, the conditions of those papers are more method-specific and their approach is quite different from the one we propose. For linear systems, persistence of excitation has thoroughly been investigated, see for example [14, 28] and the references therein.

Outline of the paper §2 presents the formal definition of DTLSSs and it formulates the major system-theoretic concepts for this system class. §3 presents a brief overview of realization theory for DTLSSs. §4 presents the main contribution of the paper.

Notation Denote by \mathbb{N} the set of natural numbers including 0. The notation described below is standard in automata theory, see [8, 4]. Consider a set X which will be called the *alphabet*. Denote by X^* the set of finite sequences of elements of X . Finite sequences of elements of X are referred to as *strings* or *words* over X . Each non-empty word w is

of the form $w = a_1 a_2 \cdots a_k$ for some $a_1, a_2, \dots, a_k \in X$. The element a_i is called the *ith letter of w* , for $i = 1, \dots, k$ and k is called the *length of w* . We denote by ϵ the *empty sequence (word)*. The length of word w is denoted by $|w|$; note that $|\epsilon| = 0$. We denote by X^+ the set of non-empty words, i.e. $X^+ = X^* \setminus \{\epsilon\}$. We denote by wv the concatenation of word $w \in X^*$ with $v \in X^*$. For each $j = 1, \dots, m$, e_j is the j th unit vector of \mathbb{R}^m , i.e. $e_j = (\delta_{1,j}, \dots, \delta_{n,j})$, $\delta_{i,j}$ is the Kronecker symbol.

2 Linear switched systems

In this section we present the formal definition of DTLSSs along with a number of relevant system-theoretic concepts for DTLSSs .

Definition 1. Recall from [19] that a discrete-time linear switched system (abbreviated by DTLSS), is a discrete-time control system of the form

$$\Sigma \begin{cases} x_{t+1} &= A_{q_t} x_t + B_{q_t} u_t \text{ and } x_0 = 0 \\ y_t &= C_{q_t} x_t. \end{cases} \quad (1)$$

Here $Q = \{1, \dots, D\}$ is the finite set of discrete modes, D is a positive integer, $q_t \in Q$ is the switching signal, $u_t \in \mathbb{R}$ is the continuous input, $y_t \in \mathbb{R}^p$ is the output and $A_q \in \mathbb{R}^{n \times n}$, $B_q \in \mathbb{R}^{n \times m}$, $C_q \in \mathbb{R}^{p \times n}$ are the matrices of the linear system in mode $q \in Q$.

Throughout the section, Σ denotes a DTLSS of the form (1). The inputs of Σ are the continuous inputs $\{u_t\}_{t=0}^\infty$ and the switching signal $\{q_t\}_{t=0}^\infty$. The state of the system at time t is x_t . Note that any switching signal is admissible and that the initial state is assumed to be zero. We use the following notation for the inputs of Σ .

Notation 1 (Hybrid inputs). Denote $\mathcal{U} = Q \times \mathbb{R}^m$.

We denote by \mathcal{U}^* (resp. \mathcal{U}^+) the set of all finite (resp. non-empty and finite) sequences of elements of \mathcal{U} . A sequence

$$w = (q_0, u_0) \cdots (q_t, u_t) \in \mathcal{U}^+, t \geq 0 \quad (2)$$

describes the scenario, when the discrete mode q_i and the continuous input u_i are fed to Σ at time i , for $i = 0, \dots, t$.

Definition 2 (State and output). Consider a state $x_{init} \in \mathbb{R}^n$. For any $w \in \mathcal{U}^+$ of the form (2), denote by $x_\Sigma(x_{init}, w)$ the state of Σ at time $t + 1$, and denote by $y_\Sigma(x_{init}, w)$ the output of Σ at time t , if Σ is started from x_{init} and the inputs $\{u_i\}_{i=0}^t$ and the discrete modes $\{q_i\}_{i=0}^t$ are fed to the system.

That is, $x_\Sigma(x_{init}, w)$ is defined recursively as follows; $x_\Sigma(x_{init}, \epsilon) = x_{init}$, and if $w =$

$v(q, u)$ for some $(q, u) \in \mathcal{U}$, $v \in \mathcal{U}^*$, then

$$x_\Sigma(x_{init}, w) = A_q x_\Sigma(x_{init}, v) + B_q u.$$

If $w \in \mathcal{U}^+$ and $w = v(q, u)$, $(q, u) \in \mathcal{U}$, $v \in \mathcal{U}^*$, then

$$y_\Sigma(x_{init}, w) = C_q x_\Sigma(x_{init}, v).$$

Definition 3 (Input-output map). *The map $y_\Sigma : \mathcal{U}^+ \rightarrow \mathbb{R}^p$, $\forall w \in \mathcal{U}^+ : y_\Sigma(w) = y(x_0, w)$, is called the input-output map of Σ .*

That is, the input-output map of Σ maps each sequence $w \in \mathcal{U}^+$ to the output generated by Σ under the hybrid input w , if started from the zero initial state. The definition above implies that the input-output behavior of a DTLSS can be formalized as a map

$$f : \mathcal{U}^+ \rightarrow \mathbb{R}^p. \quad (3)$$

The value $f(w)$ for w of the form (2) represents the output of the underlying black-box system at time t , if the continuous inputs $\{u_i\}_{i=0}^t$ and the switching sequence $\{q_i\}_{i=0}^t$ are fed to the system.

Next, we define when a general map f of the form (3) is adequately described by the DTLSS Σ , i.e. when Σ is a realization of f .

Definition 4 (Realization). *The DTLSS Σ is a realization of an input-output map f of the form (3), if f equals the input-output map of Σ , i.e. $f = y_\Sigma$.*

For the notions of observability and span-reachability of DTLSSs we refer the reader to [20, 23].

Definition 5 (Dimension). *The dimension of Σ , denoted by $\dim \Sigma$, is the dimension n of its state-space.*

Definition 6 (Minimality). *Let f be an input-output map. Then Σ is a minimal realization of f , if Σ is a realization of f , and for any DTLSS $\hat{\Sigma}$ which is a realization of f , $\dim \Sigma \leq \dim \hat{\Sigma}$.*

3 Overview of realization theory

Below we present an overview of the results on realization theory of DTLSSs along with the concept of Markov-parameters. For more details on the topic see [20]. In the sequel, Σ denotes a DTLSS of the form (1), and f denotes an input-output map $f : \mathcal{U}^+ \rightarrow \mathbb{R}^p$.

For our purposes the most important result is the one which states that a DTLSS realization of f can be computed from the Markov-parameters of f . In order to present

this result, we need to define the Markov-parameters of f formally. Denote $Q^{k,*} = \{w \in Q^* \mid |w| \geq k\}$. Define the maps $S_j^f : Q^{2,*} \rightarrow \mathbb{R}^p$, $j = 1, \dots, m$ as follows; for any $v = \sigma_1 \dots \sigma_{|v|} \in Q^*$ with $\sigma_k \in Q$, and for any $q, q_0 \in Q$,

$$S_j^f(q_0 v q) = \begin{cases} f((q_0, e_j)(q, 0)) & \text{if } v = \epsilon \\ f((q_0, e_j)(\sigma_1, 0) \dots (\sigma_{|v|}, 0)(q, 0)) & \text{if } |v| \geq 1 \end{cases} \quad (4)$$

with $e_j \in \mathbb{R}^m$ is the vector with 1 as its j th entry and zero everywhere else. The collection of maps $\{S_j^f\}_{j=1}^m$ is called the *Markov-parameters* of f .

The functions S_j^f , $j = 1, \dots, m$ can be viewed as *input responses*. The interpretation of S_j^f will become more clear after we define the concept of a *generalized convolution representation*. Note that the values of the Markov-parameters can be obtained from the values of f .

Notation 2 (Sub-word). *Consider the sequence $v = q_0 \dots q_t \in Q^+$, $q_0, \dots, q_t \in Q$, $t \geq 0$. For each $j, k \in \{0, \dots, t\}$, define the word $v_{j|k} \in Q^*$ as follows; if $j > k$, then $v_{j|k} = \epsilon$, if $j = k$, then $v_{j|j} = q_j$ and if $j < k$, then $v_{j|k} = q_j q_{j+1} \dots q_k$. That is, $v_{j|k}$ is the sub-word of v formed by the letters from the j th to the k th letter.*

Definition 7 (Convolution representation). *The input-output map f has a generalized convolution representation (abbreviated as **GCR**), if for all $w \in \mathcal{U}^+$ of the form (2), $f(w)$ can be expressed via the Markov-parameters of f as follows.*

$$f(w) = \sum_{k=0}^{t-1} S^f(q_k \cdot v_{k+1|t-1} \cdot q_t) u_k$$

where $S^f(w) = [S_1^f(w) \dots S_m^f(w)] \in \mathbb{R}^{p \times m}$ for all $w \in Q^*$.

Remark 1. *If f has a **GCR**, then the Markov-parameters of f determine f uniquely.*

The motivation for introducing **GCRs** is that existence of a **GCR** is a necessary condition for realizability by DTLSSs. Moreover, if f is realizable by a DTLSS, then the Markov-parameters of f can be expressed as products of the matrices of its DTLSS realization. In order to formulate this result more precisely, we need the following notation.

Notation 3. *Consider the collection of $n \times n$ matrices A_σ , $\sigma \in X$. For any $w \in Q^*$, the $n \times n$ matrix A_w is defined as follows. If $w = \epsilon$, then A_ϵ is the identity matrix. If $w = \sigma_1 \sigma_2 \dots \sigma_k \in X^*$, $\sigma_1, \dots, \sigma_k \in X$, $k > 0$, then*

$$A_w = A_{\sigma_k} A_{\sigma_{k-1}} \dots A_{\sigma_1}. \quad (5)$$

Lemma 1. *The map f is realized by the DTLSS Σ if and only if f has a **GCR** and for*

all $v \in Q^*$, $q, q_0 \in Q$,

$$S_j^f(q_0 v q) = C_q A_v B_{q_0} e_j, \quad j = 1, \dots, m. \quad (6)$$

Next, we define the concept of a Hankel-matrix. Similarly to the linear case, the entries of the Hankel-matrix are formed by the Markov parameters. For the definition of the Hankel-matrix of f , we will use lexicographical ordering on the set of sequences Q^* .

Remark 2 (Lexicographic ordering). Recall that $Q = \{1, \dots, D\}$. We define a lexicographic ordering \prec on Q^* as follows. For any $v, s \in Q^*$, $v \prec s$ if either $|v| < |s|$ or $0 < |v| = |s|$, $v \neq s$ and for some $l \in \{1, \dots, |s|\}$, $v_l < s_l$ with the usual ordering of integers and $v_i = s_i$ for $i = 1, \dots, l-1$. Here v_i and s_i denote the i th letter of v and s respectively. Note that \prec is a complete ordering and $Q^* = \{v_1, v_2, \dots\}$ with $v_1 \prec v_2 \prec \dots$. Note that $v_1 = \epsilon$ and for all $i \in \mathbb{N}$, $q \in Q$, $v_i \prec v_i q$.

In order to simplify the definition of a Hankel-matrix, we introduce the notion of a combined Markov-parameter.

Definition 8 (Combined Markov-parameters). A combined Markov-parameter $M^f(v)$ of f indexed by the word $v \in Q^*$ is the following $pD \times Dm$ matrix

$$M^f(v) = \begin{bmatrix} S^f(1v1), & \dots, & S^f(Dv1) \\ S^f(1v2), & \dots, & S^f(Dv2) \\ \vdots & \dots & \vdots \\ S^f(1vD), & \dots, & S^f(DvD) \end{bmatrix} \quad (7)$$

Definition 9 (Hankel-matrix). Consider the lexicographic ordering \prec of Q^* from Remark 2. Define the Hankel-matrix H_f of f as the following infinite matrix

$$H_f = \begin{bmatrix} M^f(v_1 v_1) & M^f(v_2 v_1) & \dots & M^f(v_k v_1) & \dots \\ M^f(v_1 v_2) & M^f(v_2 v_2) & \dots & M^f(v_k v_2) & \dots \\ M^f(v_1 v_3) & M^f(v_2 v_3) & \dots & M^f(v_k v_3) & \dots \\ \vdots & \vdots & \dots & \vdots & \dots \end{bmatrix},$$

i.e. the $pD \times (mD)$ block of H_f in the block row i and block column j equals the combined Markov-parameter $M^f(v_j v_i)$ of f . The rank of H_f , denoted by $\text{rank } H_f$, is the dimension of the linear span of its columns.

The main result on realization theory of DTLSSs can be stated as follows.

Theorem 1 ([20]). 1. The map f has a realization by a DTLSS if and only if f has a **GCR** and $\text{rank } H_f < +\infty$.

2. A minimal DTLSS realization of f can be constructed from H_f and any minimal DTLSS realization of f has dimension $\text{rank } H_f$.

3. A DTLSS Σ is a minimal realization of f if and only if Σ is span-reachable, observable and it is a realization of f . Any two DTLSSs which are minimal realizations of f are isomorphic².

Note that Theorem 1 shows that the knowledge of the Markov-parameters is necessary and sufficient for finding a state-space representation of f . In fact, similarly to the continuous-time case [21], we can even show that the knowledge of *finitely many* Markov-parameters is sufficient. This will be done by formulating a *realization algorithm* for DTLSSs, which computes a DTLSSs realization of f based on finitely many Markov-parameters of f .

In order to present the realization algorithm, we need the following notation.

Notation 4. Consider the lexicographic ordering \prec of Q^* and recall that $Q^* = \{v_1, v_2, \dots\}$ where $v_1 \prec v_2 \dots$. Denote by $\mathbf{N}(L)$ the number of sequences from Q^* of length at most L . It then follows that $|v_i| \leq L$ if and only if $i \leq \mathbf{N}(L)$.

Definition 10 ($H_{f,L,M}$ sub-matrices of H_f). For $L, K \in \mathbb{N}$ define the integers $I_L = \mathbf{N}(L)pD$ and $J_K = \mathbf{N}(K)mD$. Denote by $H_{f,L,K}$ the following upper-left $I_L \times J_K$ sub-matrix of H_f ,

$$\begin{bmatrix} M^f(v_1 v_1) & M^f(v_2 v_1) & \cdots & M^f(v_{\mathbf{N}(K)} v_1) \\ M^f(v_1 v_2) & M^f(v_2 v_2) & \cdots & M^f(v_{\mathbf{N}(K)} v_2) \\ \vdots & \vdots & \cdots & \vdots \\ M^f(v_1 v_{\mathbf{N}(L)}) & M^f(v_2 v_{\mathbf{N}(L)}) & \cdots & M^f(v_{\mathbf{N}(K)} v_{\mathbf{N}(L)}) \end{bmatrix}.$$

Notice that the entries of $H_{f,L,K}$ are Markov-parameters indexed by words of length at most $L + K$, i.e. $H_{f,L,K}$ is uniquely determined by $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(L+K)}$.

The promised realization algorithm is Algorithm 1, which takes as input the matrix $H_{f,N,N+1}$ and produces a DTLSS. Note that the knowledge of $H_{f,N,N+1}$ is equivalent to the knowledge of the finite sequence $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2N+1)}$ of Markov-parameters. The correctness of Algorithm 1 is stated below.

Theorem 2. If $\text{rank } H_{f,N,N} = \text{rank } H_f$, then Algorithm 1 returns a minimal realization Σ_N of f . The condition $\text{rank } H_{f,N,N} = \text{rank } H_f$ holds for a given N , if there exists a DTLSS realization Σ of f such that $\dim \Sigma \leq N + 1$.

The proof of Theorem 2 is completely analogous to its continuous-time counterpart [21]. Theorem 2 implies that if f is realizable by a DTLSS, then a minimal DTLSS realization of f is computable from finitely many Markov-parameters, using Algorithm 1. In fact, if f is realizable by a DTLSS of dimension n , then the first $\mathbf{N}(2n - 1)$ Markov-parameters $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ uniquely determines f .

²see [20] for the definition of isomorphism between DTLSSs

Algorithm 1**Inputs:** Hankel-matrix $H_{f,N,N+1}$.**Output:** DTLSS Σ_N

- 1: Let $n = \text{rank } H_{f,N,N+1}$. Choose a tuple of integers (i_1, \dots, i_n) such that the columns of $H_{f,N,N+1}$ indexed by i_1, \dots, i_n form a basis of $\text{Im} H_{f,N,N+1}$. Let \mathbf{O} be $I_N \times n$ matrix formed by these linearly independent columns, i.e. the r th column of \mathbf{O} equals the i_r th column of $H_{f,N,N+1}$. Let $\mathbf{R} \in \mathbb{R}^{n \times J_{N+1}}$ be the matrix, r th column of which is formed by coordinates of the r th column of $H_{f,N,N+1}$ with respect to the basis consisting of the columns i_1, \dots, i_n of $H_{f,N,N+1}$, for every $r = 1, \dots, J_{N+1}$. It then follows that $H_{f,N,N+1} = \mathbf{O}\mathbf{R}$ and $\text{rank } \mathbf{R} = \text{rank } \mathbf{O} = n$.
- 2: Define $\bar{\mathbf{R}} \in \mathbb{R}^{n \times J_N}$ as the matrix formed by the first J_N columns of \mathbf{R} .
- 3: For each $q \in Q$, let $\mathbf{R}_q \in \mathbb{R}^{n \times J_N}$ be such that for each $i = 1, \dots, J_N$, the i th column of \mathbf{R}_q equals the $r(i)$ th column of \mathbf{R} . Here $r(i) \in \{1, \dots, J_{N+1}\}$ is defined as follows. Consider the decomposition $i = (r-1)mD + z$ for some $z = 1, \dots, mD$ and $r = 1, \dots, \mathbf{N}(N)$. Consider the word $v_r q$ and notice that $|v_r q| \leq N+1$. Hence, $v_r q = v_d$ for some $d = 1, \dots, \mathbf{N}(N+1)$. Then define $r(i)$ as $r(i) = (d-1)mD + z$.
- 4: Construct Σ_N of the form (1) such that

$$\begin{aligned} [B_1, \dots, B_D] = \\ \text{the first } mD \text{ columns of } \mathbf{R} \end{aligned} \tag{8}$$

$$[C_1^T \ C_2^T \ \dots \ C_D^T]^T = \text{the first } pD \text{ rows of } \mathbf{O} \tag{9}$$

$$\forall q \in Q : A_q = \mathbf{R}_q \bar{\mathbf{R}}^+ \tag{10}$$

where $\bar{\mathbf{R}}^+$ is the Moore-Penrose pseudoinverse of $\bar{\mathbf{R}}$.

- 5: Return Σ_N
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The intuition behind Algorithm 1 is the following. The state-space of the DTLSS Σ_N returned by Algorithm 1 is an isomorphic copy of the space spanned by the columns of $H_{f,N,N}$. The isomorphism is determined by the matrix \mathbf{R} . The columns of B_q , $q \in Q$ are formed by the columns $(q-1)mD+1, \dots, qmD$ of the block-matrix

$$\begin{bmatrix} M^f(v_1 v_1)^T & \dots & M^f(v_1 v_{\mathbf{N}(L)})^T \end{bmatrix}^T.$$

The rows of C_q , $q \in Q$ are formed by the rows $(q-1)p+1, \dots, pq$ of $H_{f,N,N+1}$. Finally, the matrix A_q , $q \in Q$ is the matrix of a shift-like operator, which maps a block-column $\{M^f(v_j v_i)\}_{i=1}^{\mathbf{N}(L)}$ of $H_{f,N,N}$ to the block-column $\{M^f(v_j q v_i)\}_{i=1}^{\mathbf{N}(L)}$ of $H_{f,N,N+1}$.

4 Main results of the paper

The main idea behind our definition of persistence of excitation is as follows. The measured time series is persistently exciting, if from this time-series we can reconstruct the Markov-parameters of the underlying system. Note that by Theorem 2, it is enough to reconstruct *finitely many* Markov-parameters. This also means that our definition of persistence of excitation is also applicable to finite time series.

In order to present our main results, we will need some terminology.

Definition 11 (Output time-series). *For any input-output map f and for any finite input sequence $w \in \mathcal{U}^+$ we denote by $\mathbf{O}(f, w)$ the output time series induced by f and w , i.e. if w is of the form (2), then $\mathbf{O}(f, w) = \{y_t\}_{t=0}^T$, such that $y_t = f((q_0, u_0) \cdots (q_t, u_t))$ for all $t \leq T$.*

Definition 12 (Persistence of excitation). *The finite sequence $w \in \mathcal{U}^+$ is persistently exciting for the input-output map f , if it is possible to determine the Markov-parameters of f from the data $(w, \mathbf{O}(f, w))$.*

Remark 3 (Interpretation). *Theorem 2 and Algorithm 1 allow the following interpretation of persistence of excitation defined above. If w is persistently exciting, then the Markov-parameters of f can be computed from the response of f to the prefixes of w . In particular, if f admits a DTLSS realization of dimension at most n , then the Markov-parameters $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ can be computed from the data $(w, \mathbf{O}(f, w))$. The knowledge of $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ is sufficient for computing a DTLSS realization of f . Hence, persistence of excitation of w for f means that Algorithm 1 can serve as an identification algorithm for computing a DTLSS realization of f from the time-series $(w, \mathbf{O}(f, w))$. Note, however, that our definition does not depend on Algorithm 1. Indeed, if there is any algorithm which can correctly find a DTLSS realization of f from $(w, \mathbf{O}(f, w))$, then according to our definition, w is persistently exciting. Note that our definition of persistence of excitation involves only the inputs, but not the output response.*

So far we have defined the persistence of excitation for finite sequences of inputs. Next, we define the same notion for infinite sequences of inputs. To this end, we need the following notation.

Notation 5. We denote by \mathcal{U}^ω the set of infinite sequences of hybrid inputs. That is, any element $w \in \mathcal{U}^\omega$ can be interpreted as a time-series $w = \{(q_t, u_t)\}_{t=0}^\infty$. For each $N \in \mathbb{N}$, denote by w_N the sequence formed by the first N elements of w , i.e. $w_N = (q_0, u_0) \cdots (q_N, u_N)$.

Definition 13 (Asymptotic persistence of excitation). An infinite sequence of inputs $w \in \mathcal{U}^\omega$ is called asymptotically persistently exciting for the input-output map f , if the following holds. For every sufficiently large N , we can compute from $(w_N, \mathbf{O}(f, w_N))$ asymptotic estimates of the Markov-parameters of f . More precisely, for $N \in \mathbb{N}$, we can compute from $(w_N, \mathbf{O}(f, w_N))$ some matrices $\{M_N^f(v)\}_{v \in Q^*}$ such that $\lim_{N \rightarrow \infty} M_N^f(v) = M^f(v)$ for all $v \in Q^*$. When clear from the context, we will use the term persistently exciting instead of asymptotically persistently exciting.

Remark 4 (Interpretation). The interpretation of asymptotic persistence of excitation is that asymptotically persistently exciting inputs allow us to estimate a DTLSS realization of f with arbitrary accuracy. Indeed, assume that $w \in \mathcal{U}^\omega$ is asymptotically persistently exciting. Then for each N we can compute from the time-series $(w_N, \mathbf{O}(f, w_N))$ an approximation $\{M_N^f(v)\}_{v \in Q^*}$ of the Markov-parameters of f . Suppose that f is realizable by a DTLSS of dimension n and we know the indices (i_1, \dots, i_n) of those columns of $H_{f,n-1,n}$ which form a basis of the column space of $H_{f,n-1,n}$. Let $H_{f,n-1,n}^N$ be the matrix which is constructed in the same way as $H_{f,n-1,n}$, but with $M_N^f(v)$ instead of the Markov-parameters $M^f(v)$. Since $M_N^f(v)$ converges to $M^f(v)$ for all $v \in Q^*$, we get that each entry of $H_{f,n-1,n}^N$ converges to the corresponding entry of $H_{f,n-1,n}$. Modify Algorithm 1 by fixing the choice of columns to (i_1, \dots, i_n) in the first step. It is easy to see that the modified algorithm represents a continuous map from the input data (finite Hankel-matrix) to the output data (matrices of a DTLSS). For sufficiently large N , the columns of $H_{f,n-1,n}^N$ indexed by (i_1, \dots, i_n) also represent a basis of the column space of $H_{f,n-1,n}^N$. If we apply the modified Algorithm 1 to the sequence of matrices $H_{f,n-1,n}^N$, we obtain a sequence of DTLSSs $\Sigma_{n,N}$ and the parameters of $\Sigma_{n,N}$ converge to the parameters of the DTLSS Σ which we would obtain from Algorithm 1 if we applied it to $H_{f,n-1,n}$. In particular, by choosing a sufficiently large N , the parameters of $\Sigma_{n,N}$ are sufficiently close to those of Σ .

We will show that for **every** reversible DTLSS there exists **some** input which is persistently exciting. In addition, we present a class of inputs which are persistently exciting of any input-output map f realizable by a stable DTLSS.

4.1 Persistently exciting input for specific systems

In this section we present results which state that for any input-output map f which is realizable by a DTLSS, there exists a persistently exciting finite input.

Note that from (4) it follows that the Markov-parameters of f can be obtained from finitely many input-output data. However, the application of (4) implies evaluating the response of the system for different inputs, while started from a fixed initial state. In order to simulate this by evaluating the response of the system to one single input (which is then necessarily persistently exciting), one has to provide means to reset the system to its initial state. In order to be able to do so, we restrict attention to *reversible* DTLSSs.

Definition 14. A DTLSS Σ of the form (1) is reversible, if for every discrete mode $q \in Q$, the matrix A_q is invertible.

Reversible DTLSSs arise naturally when sampling continuous-time systems.

Theorem 3. Consider an input-output map f . Assume that f has a realization by a reversible DTLSS. Then there exists an input $w \in \mathcal{U}^+$ such that w is persistently exciting for f .

Sketch of the proof. The main idea behind the proof of Theorem 3 is as follows. If f admits a DTLSS realization of dimension n , then the finite sequence $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ of Markov-parameters determine all the Markov-parameters of f uniquely. Hence, in order for a finite input w to be persistently exciting for f , it is sufficient that $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ can be computed from the response $(w, \mathbf{O}(f, w))$.

Note that (4) implies that $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ can be computed from the responses of f from finitely many inputs. More precisely, $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ can be computed from $\{f(s) \mid s \in S\}$, where

$$S = \{(q_0, e_j)(\sigma_1, 0) \dots (\sigma_{|v_i|}, 0)(q, 0) \in \mathcal{U}^+ \mid q_0, q \in Q, \\ v_i = \sigma_1 \dots \sigma_{|v_i|}, j = 1, \dots, m, i = 1, \dots, \mathbf{N}(2n-1)\}.$$

Hence, if for each $s \in S$ there exists a prefix p of w such that $f(s) = f(p)$, then this w will be persistently exciting.

One way to construct such a w is to construct for each $s \in S$ an input $s^{-1} \in \mathcal{U}^+$ such that

$$\forall v \in \mathcal{U}^+ : f(ss^{-1}v) = f(v).$$

That is, the input s^{-1} neutralizes the effect of the input s . We defer the construction of the input s^{-1} to the end of the proof. Assume for the moment being that such inputs s^{-1} exist. Let $S = \{s_1, \dots, s_d\}$ be an enumeration of S . Then it is easy to see that $f(s_1 s_1^{-1} s_2) = f(s_2)$, $f(s_1 s_1^{-1} s_2 s_2^{-1} s_3) = f(s_3)$, etc. Hence, if we define

$$w = s_1 s_1^{-1} \dots s_{d-1} s_{d-1}^{-1} s_d,$$

then each $f(s)$, $s \in S$ can be obtained as a response of f to a suitable prefix of w . Hence, w is persistently exciting.

It is left to show that s^{-1} exists. Consider a reversible realization Σ of f . Then the controllable set and reachable set of Σ coincide by [7]. Hence, from any reachable state x of Σ , there exists an input $w(x)$ such that $w(x)$ drives Σ from x to zero, i.e. $x_\Sigma(x, w(x)) = 0$. For each $s \in S$, let $x(s) = x_\Sigma(0, s)$ and define $s^{-1} = w(x(s))$ as the input which drives $x(s)$ back to the initial zero state. \square

It is easy to see that Theorem 3 can be extended to any input-output map which admits a controllable DTLSS realization. However, it is not clear if every input-output map which is realizable by a DTLSS is also realizable by a controllable DTLSS. Note that the construction of the persistently exciting w from Theorem 3 requires the knowledge of a DTLSS realization of f . Below we present a subclass of input-output maps, for which the knowledge of a state-space representation is not required to construct a persistently exciting input.

Definition 15. Fix a map $\cdot^{-1} : \mathcal{U} \ni \alpha \mapsto \alpha^{-1} \in \mathcal{U}$. A input-output map f is said to be reversible with respect to the map \cdot^{-1} , if for all $\alpha \in \mathcal{U}$, $s, w \in \mathcal{U}^*$, $|sw| > 0$,

$$f(s\alpha\alpha^{-1}w) = f(sw).$$

Intuitively, f is reversible with respect to \cdot^{-1} , if the effect of any input $\alpha = (q, u)$ can be neutralized by the input α^{-1} . Such a property is not that uncommon, think for example of turning a valve on and off. For example, if f has a realization by a DTLSS Σ of the form (1), and $Q = \{1, \dots, 2K\}$ such that for each $q \in \{1, \dots, K\}$, $A_q = A_{q+K}^{-1}$, $B_q = -AB_{q+K}$, then f is reversible and $(q, u)^{-1} = (q + K, -u)$.

From the proof of Theorem 3, we obtain the following corollary.

Theorem 4. If f is reversible with respect to \cdot^{-1} , then a persistently exciting input sequence w can be constructed for f . The construction does not require the knowledge of a DTLSS state-space realization of f . If the inputs α^{-1} from Definition 15 are computable from α , then the construction of w is effective.

Proof of Theorem 4. The proof differs from that of Theorem 3 only in the definition of s^{-1} for each $s \in S$. More precisely, if f is reversible, then for each $s = (q_0, u_0) \cdots (q_t, u_t) \in S$ define

$$s^{-1} = (q_t, u_t)^{-1}(q_{t-1}, u_{t-1})^{-1} \cdots (q_0, u_0)^{-1}$$

\square

4.2 Universal persistently exciting inputs

Next, we discuss classes of inputs which are persistently exciting for all input-output maps realizable by DTLSSs.

Definition 16 (Persistence of excitation cond). *An infinite input $w = \{(q_t, u_t)\}_{t=0}^\infty \in \mathcal{U}^\omega$ satisfies PE condition, if there exists a strictly positive definite $m \times m$ matrix \mathcal{R} and a collection of strictly positive numbers $\{\pi_v\}_{v \in Q^+}$ such that for any positive integer $j \in \mathbb{N}$ and any word $v \in Q^+$*

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N u_{t+j} u_t^T \chi(q_t q_{t+1} \cdots q_{t+|v|-1} = v) &= 0 \\ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=j}^N u_{t-j} u_t^T \chi(q_{t-j} q_{t-j+1} \cdots q_{t-j+|v|-1} = v) &= 0 \\ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N u_t u_t^T \chi(q_t \cdots q_{t+|v|-1} = v) &= \pi_v \mathcal{R} \end{aligned}$$

where χ is the indicator function, i.e. $\chi(A) = 1$ if A holds and $\chi(A) = 0$ otherwise.

Remark 5 (PE condition implies rich switching). *Note that if $w \in \mathcal{U}^\omega$ satisfies the conditions of Definition 16, then*

$$\lim_{N \rightarrow \infty} \sum_{t=0}^N \chi(q_t \cdots q_{t+|v|-1} = v) = \pi_v > 0$$

for each $v \in Q^+$. This implies that any sequence of discrete modes occurs in the switching signal. Hence, our condition for persistence of excitation implies that the switching signal should be rich enough. This is consistent with many of the existing definitions of persistence of excitation for hybrid systems. The requirement that $\pi_v > 0$ for all $v \in Q^*$ is quite a strong one. At the end of this section we will discuss possible relaxations of this requirement.

Remark 6 (Relationship with stochastic processes). *Fix a probability space (Ω, \mathcal{F}, P) and consider ergodic discrete-time stochastic processes $\mathbf{u}_t : \Omega \rightarrow \mathbb{R}^m$ and $\mathbf{q}_t : \Omega \rightarrow Q$ with values in \mathbb{R}^m and Q respectively. In addition, assume the following.*

- The processes \mathbf{u}_t and \mathbf{q}_t are independent (i.e. the σ -algebras generated by $\{\mathbf{u}_t\}_{t=0}^\infty$ and by $\{\mathbf{q}_t\}_{t=0}^\infty$ are independent).
- The stochastic process \mathbf{u}_t is a colored noise, i.e. it is zero-mean, \mathbf{u}_t and \mathbf{u}_s are uncorrelated and $E[\mathbf{u}_t \mathbf{u}_t^T] = \mathcal{R} > 0$, with $E[\cdot]$ denoting the expectation operator.
- For each $v \in Q^+$, $\pi_v = P(\mathbf{q}_t \cdots \mathbf{q}_{t+|v|-1} = v) > 0$.

It then follows that almost all sample paths of $\mathbf{u}_t, \mathbf{q}_t$ satisfy the PE condition of Definition 16. That is, there exists a set $A \in \mathcal{F}$, such that $P(A) = 0$ and for all $\omega \in \Omega \setminus A$, the sequence $w = \{(q_t, u_t) = (\mathbf{q}_t(\omega), \mathbf{u}_t(\omega))\}_{t=0}^\infty$ satisfies the PE condition.

Remark 7. If \mathbf{u}_t is a white-noise Gaussian process and if the variables \mathbf{q}_t are uniformly distributed over Q (i.e. $P(\mathbf{q}_t = q) = \frac{1}{|Q|}$ and are independent from each other and from $\{\mathbf{u}_s\}_{s=0}^\infty$, then \mathbf{u}_t and \mathbf{q}_t satisfy the conditions of Remark 6 and hence almost any sample path of \mathbf{u}_t and \mathbf{q}_t satisfies the PE condition of Definition 16.

This special case also provides a simple practical way to generate inputs which satisfy the PE conditions.

We will show that input sequences which satisfy the conditions of Definition 16 are asymptotically persistently exciting for a large class of input-output maps. The main idea behind the theorem is as follows. Consider a DTLSS Σ which is realization of f , and suppose we feed a stochastic input $\{\mathbf{q}_t, \mathbf{u}_t\}$ into Σ . Then the state \mathbf{x}_t and the output response \mathbf{y}_t of Σ will also be stochastic processes. Suppose that $\{\mathbf{q}_t, \mathbf{u}_t\}$ are stochastic processes which satisfy the conditions of Remark 6. It is easy to see that

$$\mathbf{y}_t = \sum_{k=0}^t C_{\mathbf{q}_t} A_{\mathbf{q}_{t-1}} \cdots A_{\mathbf{q}_{k+1}} B_{\mathbf{q}_k} \mathbf{u}_k.$$

and hence for all $r, q \in Q$, $v \in Q^*$, $|rvq| = t + 1$,

$$\begin{aligned} E[\mathbf{y}_t \mathbf{u}_0^T \chi(\mathbf{q}_0 \cdots \mathbf{q}_t = rvq)] &= \\ \sum_{k=0}^t C_q A_v B_r E[\mathbf{u}_k \mathbf{u}_0^T \chi(\mathbf{q}_0 \cdots \mathbf{q}_t = rvq)] &= \\ C_q A_v B_r \mathcal{R} \pi_{rvq} &= S^f(rvq) \mathcal{R} \pi_{rvq}. \end{aligned} \tag{11}$$

Hence, if we know the expectations $E[\mathbf{y}_t \mathbf{u}_0^T \chi(\mathbf{q}_0 \cdots \mathbf{q}_t = rvq)]$ for all $r, q \in Q$, $v \in Q^*$, $|rvq| = t + 1$, $t > 0$, then we can find all the Markov-parameters of f , by the following formula

$$S^f(rvq) = E[\mathbf{y}_t \mathbf{u}_0^T \chi(\mathbf{q}_0 \cdots \mathbf{q}_{t+1} = rvq)] \mathcal{R}^{-1} \frac{1}{\pi_{rvq}}.$$

Hence, the problem of estimating the Markov-parameters reduces to estimating the expectations

$$E[\mathbf{y}_t \mathbf{u}_0^T \chi(\mathbf{q}_0 \cdots \mathbf{q}_t = rvq)]. \tag{12}$$

For practical purposes, the expectations in (12) have to be estimated from a sample-path of \mathbf{y}_t , \mathbf{u}_t and \mathbf{q}_t . The most natural way to accomplish this is to use the formula

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=i}^N y_{i+t} u_i^T \chi(q_i \cdots q_{i+t} = rvq) \tag{13}$$

where y_t , u_t , q_t denote the value at time t of a sample-path of \mathbf{y}_t , \mathbf{u}_t and \mathbf{q}_t respectively. Note that y_t is in fact the output of Σ at time t , if the input $\{u_i\}_{i=0}^t$ and the switching signal $\{q_i\}_{i=0}^t$ are fed to the system.

The problem with estimating (12) by (13) is that the limit (13) may fail to exist or to converge to (12).

A particular case when (13) converges to (12) is when the process $(\mathbf{y}_t, \mathbf{u}_t, \mathbf{q}_t)$ is ergodic. In that case, we can choose a sample-path (y_t, u_t, q_t) of $(\mathbf{y}_t, \mathbf{u}_t, \mathbf{q}_t)$ for which the limit in (13) equals the expectation (12) ; in fact ‘almost all’ sample paths will have this property. This means that we can choose a suitable deterministic input sequence $\{u_t\}_{t=0}^\infty$ and a switching signal $\{q_t\}_{t=0}^\infty$, such that for the resulting output $\{y_t\}_{t=0}^\infty$, the limit (13) equals the expectation (12). That is, in that case the input $w = (q_0, u_0) \cdots (q_t, u_t) \cdots$ is asymptotically persistently exciting. However, proving ergodicity of \mathbf{y}_t is not easy. In addition, even if \mathbf{y}_t is ergodic, the particular choice of the deterministic input w for which (13) equals (12) might depend on the DTLSS itself.

For this reason, instead of using the concepts of ergodicity directly, we just show that for the input sequences w which satisfy the conditions of Definition 16, the corresponding output $\{y_t\}_{t=0}^\infty$ has the property that the limit (13) exists and it equals $S^f(rvq)\mathcal{R}\pi_{rvq}$, for any input-output map f which is realizable by a l_1 -stable DTLSS. This strategy allows us to use elementary techniques, while not compromising the practical relevance of the result.

In order to present the main result of this section, we have to define the notion of l_1 -stability of DTLSSs.

Definition 17 (Stability of DTLSSs). *A DTLSS Σ of the form (1) is called l_1 -stable, if for every $x \in \mathbb{R}^n$, the series $\sum_{v \in Q^*} \|A_v x\|_2$ is convergent.*

Remark 8 (Sufficient condition for stability). *If for all $q \in Q$, $\|A_q\|_2 < \frac{1}{|Q|}$, where $\|A_q\|_2$ is the matrix norm of A_q induced by the standard Euclidean norm, then Σ is l_1 -stable.*

Remark 9 (Asymptotic stability). *If Σ is l_1 -stable, then it is asymptotically stable, in the sense that if $s_i \in Q^*$, $i > 0$ is a sequence of words such that $\lim_{i \rightarrow \infty} |s_i| = +\infty$, then $\lim_{i \rightarrow \infty} A_{s_i} x = 0$ for all $x \in \mathbb{R}^n$.*

Intuitively it is clear why we have to restrict attention to stable systems. Recall that (4) allows us to compute the Markov-parameters of f from the responses of f to finitely many inputs. In order to obtain the response of f to several inputs from the response of f to one input, one has to find means to suppress the contribution of the current state of the system to future inputs. In §4.1 this was done by feeding inputs which drive the system back to the initial state. Unfortunately, the choice of such inputs depended on the system itself. By assuming stability, we can make sure that the effect of the past state will asymptotically diminish in time. Hence, by waiting long enough, we can approximately recover the response of f to any input.

Another intuitive explanation for assuming stability is that it is necessary for the stationarity, and hence ergodicity, of the output and state processes $\mathbf{y}_t, \mathbf{x}_t$.

Equipped with the definitions above, we can finally state the main result of the section.

Theorem 5 (Main result). *If w satisfies the PE conditions of Definition 16, then w is asymptotically persistently exciting for any input-output map f which admits a l_1 -stable DTLSS realization.*

The theorem above together with Remark 7 imply that white noise input and a binary noise switching signal are asymptotically persistently exciting. The proof of Theorem 5 relies on the following technical result.

Theorem 6. *Assume that Σ is a l_1 -stable DTLSS of the form (1), and assume that w satisfies the PE conditions. Let $\{y_t\}_{t=0}^\infty$ and $\{x_t\}_{t=0}^\infty$ be the output and state response of Σ to w , i.e. $y_t = y_\Sigma(w_t)$ and $x_t = x_\Sigma(0, w_t)$. Then for all $v, \beta \in Q^*$, $r, q \in Q$*

$$\begin{aligned} & \pi_{rvq\beta} A_v B_r \mathcal{R} = \\ \lim_{N \rightarrow \infty} & \frac{1}{N} \sum_{t=0}^N x_{t+|v|+1} u_t^T \chi(t, rvq\beta) \end{aligned} \quad (14)$$

$$\begin{aligned} & \pi_{rvq\beta} C_q A_v B_r \mathcal{R} = \\ \lim_{N \rightarrow \infty} & \frac{1}{N} \sum_{t=0}^N y_{t+|v|+1} u_t^T \chi(t, rvq\beta) \end{aligned} \quad (15)$$

Here we used the following notation: for all $s \in Q^+$,

$$\chi(t, s) = \begin{cases} 1 & \text{if } s = q_t q_{t+1} \cdots q_{t+|s|-1} \\ 0 & \text{otherwise} \end{cases}$$

Informally, Theorem 6 implies that if f is realizable by a l_1 -stable DTLSS, then the limit (13) equals (12). The proof of Theorem 6 can be found in Appendix A.

Proof of Theorem 5. For each t , denote by y_t the response of f to the first t elements of w , i.e. $y_t = f((q_0, u_0) \cdots (q_t, u_t))$. For each integer $N \in \mathbb{N}$ and for each word $v \in Q^*$, define the matrix $S_N(rvq)$ as

$$S_N(rvq) = \left(\frac{1}{N} \sum_{t=0}^N y_{t+|v|+1} u_t^T \chi(t, rvq) \right) \mathcal{R}^{-1} \frac{1}{\pi_{rvq}}$$

and define the matrix $M_N(v)$ by

$$\begin{bmatrix} S_N(1v1) & \cdots & S_N(Dv1) \\ \vdots & \ddots & \vdots \\ S_N(1vD) & \cdots & S_N(DvD) \end{bmatrix}$$

From Theorem 6 it follows that

$$\lim_{N \rightarrow \infty} S_N(rvq) = S^f(rvq)$$

and hence $\lim_{N \rightarrow \infty} M_N(v) = M^f(v)$. Hence, w is indeed asymptotically persistently

exciting. □

Remark 10 (Relaxation of PE condition). *Assume that we restrict attention to input-output maps which are realizable by a l_1 -stable DTLSS of dimension at most n , and let f be such an input-output map. In this case, one can replace the conditions of Definition 16, that $\pi_v > 0$ by the condition that $\pi_s > 0$ for all $|s| \leq 2n + 1$ and still obtain asymptotically persistently exciting inputs for f .*

Indeed, consider now any $w \in \mathcal{U}^\omega$ which satisfies Definition 16 with the exception that $\pi_v > 0$ is required only for $|v| \leq 2n + 1$. Then Theorem 6 remains valid for this case (the proof remains literally the same) and from the proof of Theorem 5 we get that for all $i = 1, \dots, \mathbf{N}(2n - 1)$,

$$S^f(rv_i q) = \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{t=0}^N y_{t+|v|+1} u_t^T \chi(t, rv_i q) \right) \mathcal{R}^{-1} \frac{1}{\pi_{rv_i q}}$$

Hence, $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ can asymptotically be estimated from $(w_N, \mathbf{O}(f, w_N))$. Since the modified Algorithm 1 from Remark 4 determines a continuous map from $\{M^f(v_i)\}_{i=1}^{\mathbf{N}(2n-1)}$ to the other Markov-parameters of f , w is asymptotically persistently exciting for f .

5 Conclusions

We defined persistence of excitation for input signals of linear switched systems. We showed existence of persistently exciting input sequences and we identified several classes of input signals which are persistently exciting.

Future work includes finding less restrictive conditions for persistence of excitation and extending the obtained results to other classes of hybrid systems.

A Technical proofs

The proof of Theorem 6 relies on the following result.

Lemma 2. *With the notation and assumptions of Theorem 6, for all $v \in Q^+$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N x_t u_t^T \chi(t, v) = 0$$

The intuition behind Lemma 2 is as follows. Each x_t is a linear combination of inputs u_0, \dots, u_{t-1} . Hence, $\frac{1}{N} \sum_{t=0}^N x_t u_t^T$ can be expressed as linear combination of terms $\frac{1}{N} \sum_{t=k}^N u_{t-k} u_t^T \chi(t, s)$ for some $s \in Q^*$, $k = 1, \dots, N$. Since each such term converges to 0 as $N \rightarrow \infty$, intuitively their linear combination should converge to 0 as well. Unfortunately, the number of summands of the above increases with N . In order to deal with

this difficulty a technique similar to the M -test for double series has to be used. The assumption that Σ is l_1 -stable is required for this technique to work.

Proof of Theorem 6. We start with the proof of (14). The proof goes by induction on the length of v .

If $v = \epsilon$, then

$$\begin{aligned} \frac{1}{N} \sum_{t=0}^N x_{t+1} u_t^T \chi(t, r\beta) &= \\ \frac{1}{N} \sum_{t=0}^N (A_{q_t} x_t + B_{q_t} u_t) u_t^T \chi(t, r\beta) &= \\ \frac{1}{N} \sum_{t=0}^N A_{q_t} x_t u_t^T \chi(t, r\beta) + \frac{1}{N} \sum_{t=0}^N B_{q_t} u_t u_t^T \chi(t, r\beta). \end{aligned} \quad (16)$$

Notice $A_{q_t} x_t u_t^T \chi(t, r\beta) = A_r x_t u_t^T \chi(t, r\beta)$ and $B_{q_t} u_t u_t^T \chi(t, r\beta) = B_r u_t u_t^T \chi(t, r\beta)$. Hence,

$$\begin{aligned} \frac{1}{N} \sum_{t=0}^N A_{q_t} x_t u_t^T \chi(t, r\beta) + \frac{1}{N} \sum_{t=0}^N B_{q_t} u_t u_t^T \chi(t, r\beta) &= \\ A_r \left(\frac{1}{N} \sum_{t=0}^N x_t u_t^T \chi(t, r\beta) \right) + B_r \left(\frac{1}{N} \sum_{t=0}^N u_t u_t^T \chi(t, r\beta) \right) \end{aligned} \quad (17)$$

From the assumptions on w it follows that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N u_t u_t^T \chi(t, r\beta) = \mathcal{R} \pi_{r\beta}$$

Hence, from the PE conditions and Lemma 2 we get that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N x_{t+1} u_t^T \chi(t, r\beta) &= \\ A_r \left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^n x_t u_t^T \chi(t, r\beta) \right) + \\ + B_r \left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^n u_t u_t^T \chi(t, r\beta) \right) &= \\ A_r 0 + B_r \mathcal{R} \pi_{r\beta} = \pi_{r\beta} B_r \mathcal{R}, \end{aligned}$$

i.e. (14) holds.

Assume that (14) holds for all words of length at most L , and assume that $v = wq$, $|w| = L$ for some $w \in Q^*$ and $q \in Q$. Then by the induction hypothesis and the assumptions on

w

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N x_{t+L+2} u_t^T \chi(t, rwq\beta) = \\
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N A_q x_{t+L+1} u_t^T \chi(t, rwq\beta) + \\
& + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N B_q u_{t+L+1} u_t^T \chi(t, rwq\beta) = \\
& = A_q A_w B_r \pi_{rwq\beta} + B_r 0 = A_w B_r \mathcal{R} \pi_{rwq\beta}.
\end{aligned} \tag{18}$$

Finally, we prove (15). Notice that

$$y_{t+|v|+2} u_t^T \chi(q, t, rvq\beta) = C_q x_{t+|v|+2} u_t^T \chi(t, rvq\beta)$$

and hence by applying (14),

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N y_{t+|v|+2} u_t^T \chi(t, rvq\beta) = \\
& C_q \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N x_{t+|v|+2} u_t^T \chi(t, rvq\beta) = \\
& C_q A_v B_r \mathcal{R} \pi_{rvq\beta}.
\end{aligned}$$

□

Proof of Lemma 2. Notice that

$$\begin{aligned}
& \sum_{t=1}^N x_t u_t^T \chi(t, v) = \\
& \sum_{t=1}^N \sum_{j=1}^{t-1} A_{q_{t-1}} \cdots A_{q_j} B_{q_{j-1}} u_{j-1} u_t^T \chi(t, v) = \\
& \sum_{k=1}^{N-1} \left(\sum_{t=k}^N A_{q_{t-1}} \cdots A_{q_{t-k+1}} B_{q_{t-k}} u_{t-k} u_t^T \chi(t, v) \right) = \\
& \sum_{r \in Q} \sum_{k=0}^{N-1} \sum_{|s|=k} A_s B_r \sum_{t=k+1}^N u_{t-k-1} u_t^T \chi(t-k-1, rsv) = \\
& \sum_{i=0}^{\mathbf{N}(N)} \sum_{r \in Q} A_{v_i} B_r \sum_{t=|v_i|+1}^N u_{t-|v_i|-1} u_t^T \chi(t-|v_i|-1, rv_i v).
\end{aligned}$$

In the last step we used the lexicographic ordering of Q^* from Remark 2. It then follows

that

$$\begin{aligned} \frac{1}{N} \sum_{t=1}^N x_t u_t^T \chi(t, v) = \\ \sum_{r \in Q} \sum_{i=0}^{\mathbf{N}(N)} A_{v_i} B_r \frac{1}{N} \sum_{t=|v_i|+1}^N u_{t-|v_i|-1} u_t^T \chi(t - |v_i| - 1, r v_i v). \end{aligned}$$

Define

$$\begin{aligned} b_{i,N}^r &= \frac{1}{N} \sum_{t=|v_i|+1}^N u_{t-|v_i|-1} u_t^T \chi(t - |v_i| - 1, r v_i v) \\ a_{i,N}^r &= A_{v_i} B_r b_{i,N}^r. \end{aligned}$$

Then the statement of the lemma can be shown by showing that for all $r \in Q$, $i = 1, 2, \dots$,

$$\lim_{N \rightarrow \infty} \sum_{k=0}^{\mathbf{N}(N)} a_{i,N}^r = 0.$$

To this end, notice from the PE conditions that

$$\begin{aligned} \lim_{N \rightarrow \infty} a_{i,N}^r &= \\ A_{v_i} B_r \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=k+1}^N u_{t-k-1} u_t^T \chi(t - k - 1, r v_i v) &= 0. \end{aligned}$$

Moreover, for a fixed N and k , we can get the following estimate

$$\|a_{i,N}^r\|_2 \leq \|A_{v_i} B_r\|_2 \|b_{i,N}^r\|_2.$$

If we can show that $\|b_{i,N}^r\|_2$ is bounded by a number K , then we get that

$$\|a_{i,N}^r\|_2 \leq \|A_{v_i} B_r\|_2 K.$$

The latter inequality is already sufficient to finish the proof. Indeed, let $D_i^r = \|A_{v_i} B_r\|_2 K$ and notice from the l_1 -stability assumption on the realization Σ that

$$\sum_{i=1}^{\infty} D_i^r = K \sum_{v \in Q^*} \|A_v B_r\|_2$$

is convergent. Hence, we get that for every $\epsilon > 0$ there exists a I_ϵ such that

$$\sum_{i=I_\epsilon+1}^{\infty} D_i^r < \epsilon/2.$$

For every $N > I_\epsilon$,

$$\begin{aligned} \left\| \sum_{i=1}^{\mathbf{N}(N)} a_{i,N}^r \right\|_2 &= \left\| \sum_{i=1}^{I_\epsilon} a_{i,N}^r + \sum_{i=I_\epsilon+1}^{\mathbf{N}(N)} a_{i,N}^r \right\|_2 \leq \\ &\sum_{i=1}^{I_\epsilon} \|a_{i,N}^r\|_2 + \sum_{i=I_\epsilon+1}^{\mathbf{N}(N)} D_i^r < \sum_{i=1}^{I_\epsilon} \|a_{i,N}^r\|_2 + \epsilon/2. \end{aligned}$$

Since $\lim_{N \rightarrow \infty} a_{i,N}^r = 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for all $N > N_\epsilon$, $\|a_{i,N}^r\|_2 < \frac{\epsilon}{2I_\epsilon}$. Define \hat{N}_ϵ to be an integer such that $\hat{N}_\epsilon > N_\epsilon$ and $\mathbf{N}(\hat{N}_\epsilon) > I_\epsilon$. Then for every $N > \hat{N}_\epsilon$, $\mathbf{N}(N) \geq \mathbf{N}(\hat{N}_\epsilon) > I_\epsilon$ and

$$\begin{aligned} \left\| \sum_{i=1}^{\mathbf{N}(N)} a_{i,N}^r \right\|_2 &\leq \sum_{i=1}^{I_\epsilon} \|a_{i,N}^r\|_2 + \epsilon/2 < \\ I_\epsilon \frac{\epsilon}{2I_\epsilon} + \epsilon/2 &= \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

In other words, $\lim_{N \rightarrow 0} \sum_{i=1}^{\mathbf{N}(N)} a_{i,N}^r = 0$.

It is left to show that $\|b_{i,N}^r\|_2 \leq K$ for some $K > 0$ and for all $i = 1, 2, \dots, r \in Q$.

$$\begin{aligned} \|b_{i,N}^r\|_2 &\leq \left\| \frac{1}{N} \sum_{t=|v_i|+1}^N u_{t-|v_i|-1} u_t^T \chi(t - |v_i| - 1, r v_i v) \right\|_2 \leq \\ \left\| \frac{1}{N} \sum_{t=|v_i|+1}^N u_{t-|v_i|-1} u_t^T \chi(t - |v_i| - 1, r v_i v) \right\|_F &= \\ \left[\sum_{i,j=1}^m \frac{1}{N^2} \left\{ \sum_{t=|v_i|+1}^N (u_{t-|v_i|-1})_i \chi(t - |v_i| - 1, r v_i v) (u_t)_j \right\}^2 \right]^{1/2}. \end{aligned} \tag{19}$$

where $\|\cdot\|_F$ denotes the matrix Frobenius-norm, and $\|\cdot\|_2$ denotes the matrix norm induced by the Euclidean norm. The application of the Cauchy-Schwartz inequality to $(\sum_{t=|v_i|+1}^N (u_{t-|v_i|-1})_i \chi(t - |v_i| - 1, r v_i v) (u_t)_j)^2$ leads to

$$\begin{aligned} \left[\sum_{t=|v_i|+1}^N (u_{t-|v_i|-1})_i \chi(t - |v_i| - 1, r v_i v) (u_t^T)_j \right]^2 &\leq \\ \left(\sum_{t=|v_i|+1}^N (u_{t-|v_i|-1})_i^2 \chi(t - |v_i| - 1, r v_i v) \right) &\left(\sum_{t=|v_i|}^N (u_t)_j^2 \right). \end{aligned} \tag{20}$$

Notice that $(u_{t-|v_i|-1})_i^2 \chi(t - |v_i| - 1, r v_i v) \leq (u_{t-|v_i|-1})_i^2$, since $\chi(t - |v_i| - 1, r v_i v) \in [0, 1]$.

Hence,

$$\begin{aligned} & \sum_{t=|v_i|+1}^N (u_{t-|v_i|-1})_i^2 \chi(t-|v_i|-1, rv_i v) \leq \\ & \leq \sum_{t=|v_i|+1}^N (u_{t-|v_i|-1})_i^2 \leq \sum_{t=0}^N (u_t)_i^2. \end{aligned}$$

Similarly,

$$\sum_{t=|v_i|+1}^N (u_t)_j^2 \leq \sum_{t=0}^N (u_t)_j^2.$$

Combining these remarks with (20), we obtain

$$\begin{aligned} & \left[\frac{1}{N^2} \sum_{t=|v_i|+1}^N (u_{t-|v_i|-1})_i \chi(t-|v_i|-1, rv_i v) (u_t^T)_j \right]^2 \\ & \leq \left(\frac{1}{N} \sum_{t=0}^N (u_t)_i^2 \right) \left(\frac{1}{N} \sum_{t=0}^N (u_t)_j^2 \right). \end{aligned} \tag{21}$$

Notice that $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=0}^N (u_t)_i^2 = \mathcal{R}_{ii}$ and hence $\frac{1}{N} \sum_{t=0}^N (u_t)_i^2$ is bounded from above by some positive number K_i . Using this fact and by substituting (21) into (19), we obtain

$$\|b_{i,N}^r\|_2 \leq \left(\sum_{i,j=1}^m K_i K_j \right)^{1/2}.$$

Hence, if we set $K = \sum_{i,j=1}^m K_i K_j$, then $\|b_{i,N}^r\|_2 \leq K$, which is what had to be shown. \square

References

- [1] L. Bako. Identification of switched linear systems via sparse optimization. *Automatica*, doi:10.1016/j.automatica.2011.01.036, 2011.
- [2] L. Bako, G. Mercre, and S. Lecoeuche. Online structured subspace identification with application to switched linear systems. *International Journal of Control*, 82:1496–1515, 2009.
- [3] L. Bako, G. Mercre, R. Vidal, and S. Lecoeuche. Identification of switched linear state space models without minimum dwell time. In *IFAC Symposium on System Identification, Saint Malo, France*, 2009.
- [4] Samuel Eilenberg. *Automata, Languages and Machines*. Academic Press, New York, London, 1974.

- [5] G. Ferrari-Trecate, M. Musellu, D. Liberati, and M. Morari. A clustering technique for the identification of piecewise-affine systems. *Automatica*, 39:205–217, 2003.
- [6] E.B. Fox. *Bayesian Nonparametric Learning of Complex Dynamical Phenomena*. Ph.D. thesis, MIT, Cambridge, MA, 2009.
- [7] S.S. Ge, Zhendong Sun, and T.H. Lee. Reachability and controllability of switched linear discrete-time systems. *IEEE Trans. Automatic Control*, 46(9):1437 – 1441, 2001.
- [8] F. Gécseg and I Peák. *Algebraic theory of automata*. Akadémiai Kiadó, Budapest, 1972.
- [9] Y. Hashambhoy and R. Vidal. Recursive identification of switched ARX models with unknown number of models and unknown orders. In *IEEE Conference on Decision and Control*, 2005.
- [10] A. Hiskens. Identifiability of hybrid system models. In *Proceedings of the IEEE International Conference on Control Applications, Anchorage, AK*, 2000.
- [11] A. Juloski. *Observer Design and Identification Methods for Hybrid Systems: Theory and Experiments*. PhD thesis, Eindhoven University of Technology, 2004.
- [12] A. Juloski, W.P.M.H. Heemels, G. Ferrari-Trecate, R. Vidal, S. Paoletto, and J. Niessen. Comparison of four procedures for the identification of hybrid systems. In *Hybrid Systems: Computation and Control*, volume 3414 of *LNCS*, pages 354–369. Springer-Verlag, Berlin, 2005.
- [13] A. Juloski, S. Weiland, and M. Heemels. A bayesian approach to identification of hybrid systems. In *Proceedings of 43rd IEEE Conference on Decision and Control*, 2004.
- [14] L. Ljung. *System Identification: Theory for the user (2nd Ed.)*. PTR Prentice Hall., Upper Saddle River, USA, 1999.
- [15] Y. Ma and R. Vidal. A closed form solution to the identification of hybrid ARX models via the identification of algebraic varieties. In *Hybrid Systems: Computation and Control*, pages 449–465, 2005.
- [16] Yi Ma and R. Vidal. Identification of deterministic switched arx systems via identification of algebraic varieties. In *Hybrid Systems: Computation and Control*, volume 3414 of *Lecture Notes in Computer Science*, pages 449 – 465, 2005.
- [17] S. Paoletti, A. Juloski, G. Ferrari-Trecate, and R. Vidal. Identification of hybrid systems: A tutorial. *European Journal of Control*, 13(2-3):242 – 260, 2007.

- [18] S. Paoletti, J. Roll, A. Garulli, and A. Vicino. Input/output realization of piecewise affine state space models. In *46th IEEE Conf. on Dec. and Control*, 2007.
- [19] M. Petreczky, L. Bako, and J.H. van Schuppen. Identifiability of discrete-time linear switched systems. In *Hybrid Systems: Computation and Control*. ACM, 2010.
- [20] Mihaly Petreczky, Laurent Bako, and Jan H. van Schuppen. Realization theory for discrete-time linear switched systems. Technical report, ArXiv, 2011.
- [21] Mihaly Petreczky and Jan H. van Schuppen. Partial-realization of linear switched systems: A formal power series approach. Technical Report arXiv:1010.5160v1, ArXiv, 2010. Available at <http://arxiv.org/abs/1010.5160v1>.
- [22] J. Roll, A. Bemporad, and L. Ljung. Identification of piecewise affine systems via mixed-integer programming. *Automatica*, 40(1):37–50, 2004.
- [23] Zhendong Sun and Shuzhi S. Ge. *Switched linear systems : control and design*. Springer, London, 2005.
- [24] V. Verdult and M. Verhaegen. Subspace identification of piecewise linear systems. In *Proc.Conf. Decision and Control*, 2004.
- [25] R. Vidal. Recursive identification of switched ARX systems. *Automatica*, 44(9):2274 – 2287, 2008.
- [26] R. Vidal, A. Chiuso, and S. Sastry. Observability and identifiability of jump linear systems. In *Proc. IEEE Conf. Dec. and Control*, pages 3614 – 3619, 2002.
- [27] R. Vidal, S. Sastry, and A. Chiuso. Observability of linear hybrid systems. In *Hybrid Systems: Computation and Control*, 2003.
- [28] Jan C. Willems, Paolo Rapisarda, Ivan Markovsky, and Bart L.M. De Moor. A note on persistency of excitation. *Systems & Control Letters*, 54(4):325 – 329, 2005.